QIS KEY CONCEPTS for HIGH SCHOOL MATHEMATICS

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FRAMEWORK FOR K-12 QUANTUM EDUCATION

The world is in the midst of a second quantum revolution due to our ability to exquisitely control quantum systems and harness them for applications in quantum computing, communications, and sensing. Quantum information science (QIS) is an area of STEM that makes use of the laws of quantum physics for the storage, transmission, manipulation, processing, or measurement of information.

After the passage of the US National Quantum Initiative Act in December 2018 [1], the National Science Foundation and the White House Office of Science and Technology Policy (WHOSTP) assembled an interagency working group and subsequently facilitated a workshop titled "Key Concepts for Future Quantum Information Science Learners" that focused on identifying core concepts essential for helping pre-college students engage with QIS. The output of this workshop was intended as a starting point for future curricular and educator activities [2-4] aimed at K-12 and beyond. Helping pre-college students learn the QIS Key Concepts could effectively introduce them to the Second Quantum Revolution and inspire them to become future contributors and leaders in the growing field of QIS spanning quantum computing, communication, and sensing.

The framework for K-12 quantum education outlined here is an expansion of the original QIS Key Concepts, providing a detailed route towards including QIS topics in K-12 physics, chemistry, computer science and mathematics classes. The framework will be released in sections as it is completed for each subject.

As QIS is an emerging area of science connecting multiple disciplines, content and curricula developed to teach QIS should follow the best practices. The K-12 quantum education framework is intended to provide some scaffolding for creating future curricula and approaches to integrating QIS into physics, computer science, mathematics, and chemistry (mathematics and chemistry are not yet complete). The framework is expected to evolve over time, with input from educators and educational researchers.

WHY QUANTUM EDUCATION AT THE K-12 LEVEL?

Starting quantum education in K-12 provides a larger, more diverse pool of students the opportunity to learn about this exciting field so that they can become the future leaders in this rapidly growing field. This is especially important because over the past century during which the first quantum revolution unfolded, the quantum-related fields have lacked gender, racial, and ethnic diversity. We must tap into the talents of students from diverse demographic groups in order to maintain our leadership in science and technology. Early introduction to quantum science can include information on applications and societal relevance, which will hopefully spark excitement and lead more students into later coursework and careers in STEM. Also, starting early with a conceptual, intuitive approach that doesn't rely on advanced mathematics will likely increase quantum awareness with more students, even those who do not pursue a career in QIS. In the long term, this will potentially improve public perception of QIS, moving it out of the weird, spooky, incomprehensible, unfamiliar realm.

WHAT ARE SOME CONSIDERATIONS TO TAKE INTO ACCOUNT WHEN INTRODUCING QIS INTO THE K-12 CLASSROOM?

As an emerging field that has traditionally been the realm of advanced undergraduate and graduate study with an aura of complexity, educators designing and delivering curriculum should keep the following in mind when integrating QIS into their classrooms.

1. Because existing materials in QIS are designed for more advanced students, the materials need to be adjusted to be age-appropriate for and build on prior knowledge of target students. As new educational research and data on implementation come in, the materials will change and improve over time.
2. Because the area may be intimidating, and there is no expectation in college that students have already learned this, motivational goals such as higher self-efficacy and a sense of belonging and identity [5-11] should be on equal footing with technical goals. Therefore, classrooms should focus on the following considerations:
   - Maintain a supportive atmosphere that encourages questions and exploration
   - Offer collaborative, exploratory activities
   - Offer a low-stakes educational setting (e.g. little time pressure without aggressive testing)
   - When relevant to the STEM subject, employ a learning cycle approach to develop models of quantum systems and phenomena, plan and carry out investigations to test their models, analyze and interpret data, obtain, evaluate, and communicate their findings

LEARNING TRAJECTORY OF INTEGRATED MATHEMATICS & QIS KEY CONCEPTS

1. **Method for creating the learning trajectory:**

   For each key concept, the group identified the mathematics required to understand the concept presented. While there are a total of nine QIS Key Concepts, we only included the subset that had clear mathematical content covered in middle school and beyond in the framework. Some key concepts include multiple concepts, so if a single key concept has multiple individual applications of mathematics, it is included multiple times.

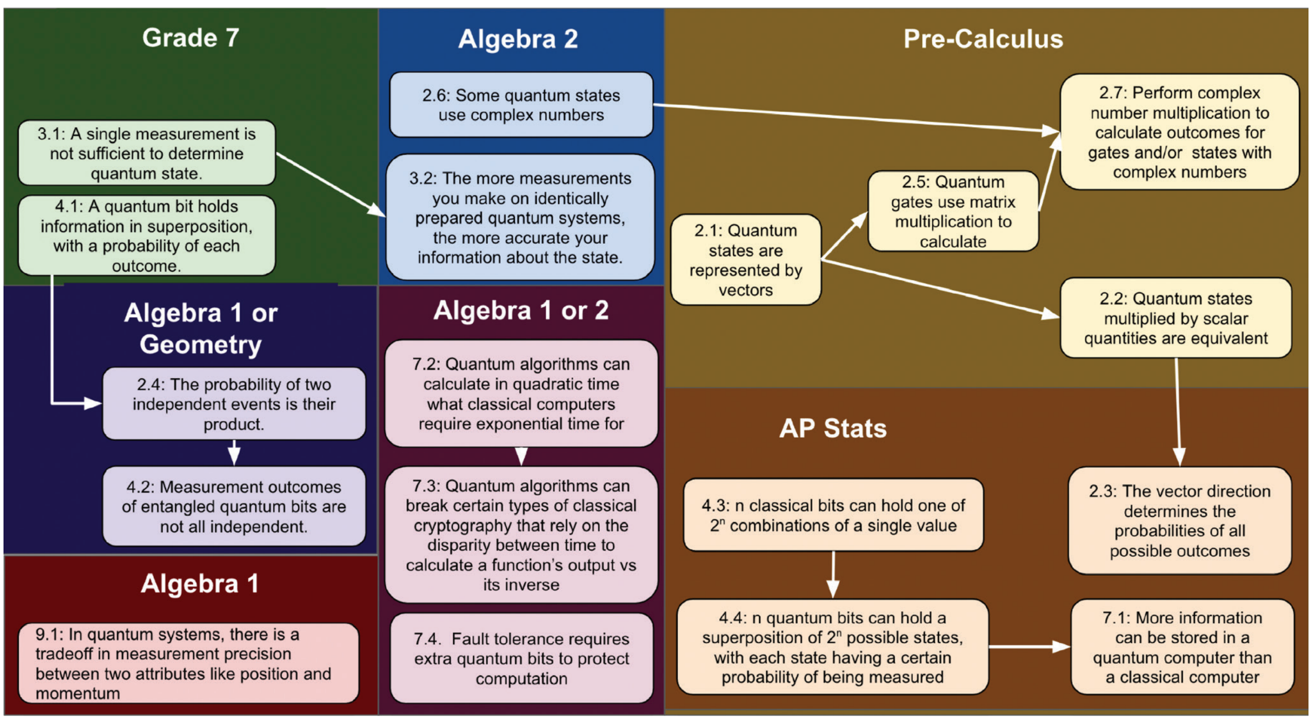

Figure 1: *Learning Trajectory of QIS Key Concepts*, from a mathematical perspective. The color indicates the course in which it would typically be taught. The number corresponds to the associated integrated learning goal below.

Once the mathematical concepts for each key concept were identified, the group identified the **Common Core State Standard** associated with the mathematical concept as well as the **course** in which that content is typically taught. (The Common Core State Standards for Mathematics are available here: https://www.nctm.org/ccssm/)

Individual courses were then identified that commonly teach that common core state standard. However, there were two circumstances in which multiple courses were identified for the same learning goal. First, not all states and school districts cover the same standards in the same courses, so the same standard might be taught in different courses (e.g. Algebra 1 or Geometry). Second, some concepts are introduced in one course and might be covered in more depth in a follow-on course, so instructors could choose to cover the Quantum concept in either course (e.g. Algebra 1 or 2).

To create relationships between different concepts, we considered whether the mathematics or QIS material being covered in one key concept was then used in a subsequent one. In the learning trajectory, there are two kinds of learning goals, both of which we had to consider.

a) Mathematical concepts that will be used in quantum later (e.g. the probability of two independent events is their product).

b) Applications of mathematical concepts to quantum (e.g. entangled qubits' outcomes are not independent).

2. **Presentation:** This document has several pieces of information associated with each learning goal of the learning trajectory:
   a) QIS Key Concept
   b) Relationship of Key Concept to mathematical concept
   c) Common Core State Standard associated with the mathematical concept
   d) Course that in which the concept is typically taught
   e) Relationship between this learning goal and any pre-requisite learning goals

QIS K-12 Key Concepts Math Focus Group

The purpose of the QIS K-12 Key Concepts Math Focus Group was to create a document that would be useful to curriculum developers and teachers, providing guidance about places where high school mathematics learning goals can be satisfied at the same time as content in the QIS K-12 Key Concepts.

The focus group brought together experts, educators familiar with teaching and research of mathematics concepts at high school and/or university levels. The members were:

Eugenie Alvares, Simpson Academy for Young Women, Chicago, IL

John Brown, Lake View High School, Chicago, IL

* Diana Franklin, University of Chicago, Chicago, IL

Kit Golan, Lesley University, Cambridge, MA

Maajida Murdock, Baltimore County Public Schools, Baltimore, MD

* Brent Yen, University of Chicago, Chicago, IL

* *Designates working group leads, conveners, and/or framework editors.*

The result is a set of activities that both teach mathematics skills typically taught in high school as well as very early QIS concepts. These are not meant to be full activities or lesson plans - different instructors may use different languages, choose to go into different amounts of depth, etc. We hope, however, that by providing examples of synergistic activities, curriculum developers will be able to use these (and their own ideas) to create either individual activities or sequences of activities that build math skills and knowledge while, at the same time, introducing students to some basic QIS concepts.

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1. QUANTUM INFORMATION SCIENCE

KEY CONCEPT

Quantum information science (QIS) exploits quantum principles to transform how information is acquired, encoded, manipulated, and applied. Quantum information science encompasses quantum computing, quantum communication, and quantum sensing, and spurs other advances in science and technology.

a. Quantum information science employs quantum mechanics, a well-tested theory that uses the mathematics of probability, vectors, algebra, trigonometry, complex numbers, and linear transformations to describe the physical world.

b. Quantum information science combines information theory and computer science, following the laws of quantum mechanics, to process information in fundamentally new ways.

c. Quantum information science has already produced and enhanced high-impact technologies such as the Global Positioning System (GPS), which depends on the extreme precision of atomic clocks. based on the quantum states of atoms.

NOTE

The definition of QIS could be discussed as a precursor to any of the other concepts, or in discussions of career opportunities in the field.

2. QUANTUM STATE

KEY CONCEPT

A **quantum state** is a mathematical representation of a physical system, such as an atom, and provides the basis for processing quantum information.

a. Quantum states are represented by directions or vectors in an abstract space.

b. (1) The direction of the quantum state vector determines the probabilities of all of the possible outcomes of a set of measurements. (2) Quantum manipulations in the physical world follow vector operations, incorporating complex numbers and negative values. (3) This captures a behavior of physical quantum systems that cannot be described solely by the arithmetic of probability.

c. Quantum systems are fragile. For instance, measurement almost always disturbs a quantum system in a way that cannot be ignored. This fragility influences the design of computational algorithms and communication and sensing protocols.

HS Mathematics 2a.1 QUANTUM STATE

DESCRIPTION

The QIS key concept is the fact that quantum states are represented by directions or vectors in an abstract space. This QIS key concept matches with the first Common Core math standard addressing vector and matrix quantities, in which students learn what a vector is and how to represent it.

LEARNING OUTCOME

Quantum states are represented by vectors

(HS.Math.2a.1)

STANDARDS ALIGNMENT

Common Core State Standard:

- (HS.N.VM.1) Recognize vector quantities as having both magnitude and direction. Represent vector quantities by directed line segments, and use appropriate symbols for vectors and their magnitudes

COURSE

- Pre-calculus

HS Mathematics 2a.2 QUANTUM STATE

DESCRIPTION

Quantum states are determined only by the direction, not the magnitude, of their quantum state vector. Therefore, multiplying a quantum state vector by any nonzero scalar results in a vector that represents the same quantum state. For example, the two vectors [1, 1] and [2, 2] represent the same quantum state - their directions are the same, with only their magnitudes being different. Here are two consequences of this fact:

a) A quantum state is meaningfully represented by a unit vector, which is a vector of magnitude 1. *(The reason why quantum states are properly represented by unit vectors is discussed in HS.Math.2b.1.)* Any vector that does not have magnitude equal to 1 can be transformed to a unit vector that represents the same quantum state by multiplying it by an appropriate scalar. For example, the vector [3, 4] can be transformed to the unit vector [3/5, 4/5] by multiplying the vector by the scalar 1/5.

b) A quantum state has multiple representations as a vector. It is often simpler and more convenient to choose to represent a quantum state by a vector that does not have magnitude 1. For example, there is a commonly used quantum state that is represented by the unit vector [1/√2, 1/√2]. It is very common to omit the 1√2 scalar and represent this quantum state by the vector [1, 1], because this is a simpler way of expressing the state that has two equal components.

The representation of quantum states as vectors provides an application for the mathematical concept of multiplying vectors with scalars.

LEARNING OUTCOME

Multiplying a quantum state by a scalar quantity results in an equivalent quantum state.; the meaningful vector length is always 1.

(HS.Math.2a.2)

STANDARDS ALIGNMENT

Common Core State Standard:

- (HS.N.VM.5) Multiply a vector by a scalar

COURSE

- Pre-calculus

HS Mathematics 2b1.1

QUANTUM STATE

DESCRIPTION

The direction, not magnitude, of the vector determines its state, as expressed in LG.2.2. The reason this is true is due to the following two facts:

a) Each component in the vector represents the probability amplitude of a single possible measurement outcome. The probability of a measurement outcome is given by the square a^2 of the corresponding probability amplitude, *a*.

b) The magnitude of the vector is calculated by computing the square root of the sum of the squares of its components. The sum of the probabilities of all possible outcomes of any scenario is always 1, so the meaningful vector magnitude is always 1.

Therefore, the direction of the vector, with the magnitude of the vector scaled to 1, provides the probabilities of all possible outcomes. This combines knowledge of direction and magnitude of vectors (LG.2.1 and LG.2.2) with basic probability concepts.

- Optional for students who are studying complex numbers: A probability amplitude can be a complex number (See learning goal LG.2.6). In this case, it is necessary to modify the first fact listed above: The probability of a measurement outcome is given by the square $|a|^2$ of the absolute value of the corresponding probability amplitude, *a*.

LEARNING OUTCOME

The quantum state vector determines the probabilities of all of the possible outcomes of a set of measurements.

(HS.Math.2b1.1)

STANDARDS ALIGNMENT

Common Core State Standard:

- (HS.S.MD.3) Develop a probability distribution for a random variable defined for a sample space in which theoretical probabilities can be calculated; find the expected value. For example, find the theoretical probability distribution for the number of correct answers obtained by guessing on all five questions of a multiple-choice test where each question has four choices, and find the expected grade under various grading schemes.

COURSE

- AP Statistics

HS Mathematics 2b1.2

QUANTUM STATE

DESCRIPTION

The probability of a measurement outcome of a two-qubit system can be computed using the same method described in the previous learning goal LG.2.3: Calculate the square of the absolute value of the corresponding probability amplitude. If two qubits are independent, the probabilities of each two-qubit measurement outcome is the product of the probabilities of the measurement outcomes for each individual qubit. When expressing the quantum state vector of an independent two-qubit system, each of the four probability amplitudes will be a product of probability amplitudes from the two individual qubits.

LEARNING OUTCOME

The probability of a combination of independent qubits' outcomes is the product of their individual probabilities

(HS.Math.2b1.2)

STANDARDS ALIGNMENT

Common Core State Standard:

- (HS.S.CP.2) Understand that two events A and B are independent if the probability of A and B occurring together is the product of their probabilities, and use this characterization to determine if they are independent.

COURSE

- Algebra
- Geometry

HS Mathematics 2b2.1

QUANTUM STATE

DESCRIPTION

Quantum states are represented as vectors (LG.2.1), whereas quantum gates are represented as matrices. To calculate the effect of applying a particular gate to a particular quantum state, one multiplies the vector by the matrix. The result of the operation is a vector — the resulting quantum state. Calculating the operation of quantum gates applied to quantum states provides an application for the mathematical concept of multiplying a vector by a matrix.

LEARNING OUTCOME

Quantum gates use matrix multiplication to calculate.

(HS.Math.2b2.1)

STANDARDS ALIGNMENT

Common Core State Standard:

- (HS.N.VM.11) Multiply a vector (regarded as a matrix with one column) by a matrix of suitable dimensions to produce another vector. Work with matrices as transformations of vectors.

COURSE

- Pre-calculus

HS Mathematics 2b2.2

QUANTUM STATE

DESCRIPTION

Quantum states have three components which are expressed using the relative magnitudes of the two vector components, a negative sign, and the complex number *i*. Introducing this third component matches (the complex number *i*) corresponds to mathematics standards introducing complex numbers, their representation with *i*, and the significance of squaring *i*.

LEARNING OUTCOME

Some quantum states use complex numbers.

(HS.Math.2b2.2)

STANDARDS ALIGNMENT

Common Core State Standard:

- (HS.N.CN.1) Know there is a complex number i such that $i^2 = -1$, and every complex number has the form a + bi with a and b real.

COURSE

- Algebra 2

HS Mathematics 2b2.3

QUANTUM STATE

DESCRIPTION

The vectors representing quantum states and the matrices representing gate operations can include complex numbers. Calculating the effects of gate operations requires matrix multiplication, which in turn requires the multiplication of complex numbers.

LEARNING OUTCOME

Perform complex number multiplication to calculate outcomes for gates and/or states with complex numbers

(HS.Math.2b2.3)

STANDARDS ALIGNMENT

Common Core State Standard:

- (HS.N.CN.2) Use the relation i2 = –1 and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.

COURSE

- Pre-calculus

3. QUANTUM MEASUREMENT

KEY CONCEPT

Quantum applications are designed to carefully manipulate fragile quantum systems without observation to increase the probability that the final **measurement** will provide the intended result.

a. A measurement is an interaction with the quantum system that transforms a state with multiple possible outcomes into a “collapsed” state that now has only one outcome: the measured outcome. *(See section on qubits)*

b. A quantum state determines the probability of the outcome of a single quantum measurement, but one outcome rarely reveals complete information about the system.

c. Repeated measurements on identically prepared quantum systems are required to determine more complete information about the state.

d. Because of the limitations of quantum measurement (providing only partial information and disturbing the system), quantum states cannot be copied or duplicated.

HS Mathematics 3b.1 QUANTUM MEASUREMENT

DESCRIPTION

For any process which has a probability associated with it, multiple data points are required. Measurements of quantum states are such a process. In 6th grade, students transition from measurements with deterministic outcomes (e.g. the length of the table, one student's age) to data sets that involve variation (e.g. average height of students in a class). In 7th grade, they further transition to probabilistic events, which matches with quantum systems. Flipping a coin and rolling dice are typical examples, which are also used to explain quantum systems.

LEARNING OUTCOME

A single measurement is not sufficient to determine a quantum state.

(HS.Math.3b.1)

STANDARDS ALIGNMENT

Common Core State Standard:

- (7.SP.6) Approximate the probability of a chance event by collecting data on the chance process that produces it and observing its long-run relative frequency, and predict the approximate relative frequency given the probability. *For example, when rolling a number cube 600 times, predict that a 3 or 6 would be rolled roughly 200 times, but probably not exactly 200 times.*

COURSE

- Grade 7 Math

HS Mathematics 3c.1 QUANTUM MEASUREMENT

DESCRIPTION

In order to determine the state of a quantum system, many measurements must be taken on the quantum system in the same state. This is because of the statistical nature of quantum states and the fact that states cannot be measured directly. Additionally, since a measurement alters the state, the system needs to be prepared in the same state prior to measurement each time. Likewise, in statistics, determining whether a model matches a known statistical state requires the same methodology. Studying the features of quantum states offer another way of coming at the same statistical concept, and allows students to further apply this concept to a physical example.

LEARNING OUTCOME

The more measurements you make on identically prepared quantum systems, the more accurate your information about the state.

(HS.Math.3c.1)

STANDARDS ALIGNMENT

Common Core State Standard:

- (HS.S.IC) Decide if a specified model is consistent with results from a given data-generating process, e.g., using simulation. *For example, a model says a spinning coin falls heads up with probability 0.5. Would a result of 5 tails in a row cause you to question the model?*

COURSE

- Algebra 2

4. QUBITS

KEY CONCEPT

The **quantum bit**, or **qubit**, is the fundamental unit of quantum information, and is encoded in a physical system, such as polarization states of light, energy states of an atom, or spin states of an electron.

a. Unlike a classical bit, each qubit can represent information in a superposition, or vector sum that incorporates two mutually exclusive quantum states.

b. At a particular moment in time, a set of n classical bits can exist in only one of 2^n possible states, but a set of n qubits can exist in a superposition of all of these states. This capability allows quantum information to be stored and processed in ways that would be difficult or impossible to do classically. *(See section on quantum computing)*

c. Multiple qubits can also be entangled, where the measurement outcome of one qubit is correlated with the measurement outcomes of the others.

HS Mathematics 4a.1

QUBITS

DESCRIPTION

Qubits store information in the form of quantum states. A quantum state expresses the probabilities of every possible measurement outcome. The very beginning of this concept occurs in 7th grade, when students learn how to interpret probabilities between 0 and 1. The idea of superposition of quantum states provides an application where students can explore the concept of probability.

LEARNING OUTCOME

A quantum bit holds information in superposition, with a probability of each outcome.

(HS.Math.4a.1)

STANDARDS ALIGNMENT

Common Core State Standard:

- (7.SP.5) Understand that the probability of a chance event is a number between 0 and 1 that expresses the likelihood of the event occurring. Larger numbers indicate greater likelihood. A probability near 0 indicates an unlikely event, a probability around 1/2 indicates an event that is neither unlikely nor likely, and a probability near 1 indicates a likely event.

COURSE

- Grade 7 Math

HS Mathematics 4b.1

QUBITS

DESCRIPTION

To understand the difference in storage between classical and quantum systems, we first need to be able to calculate how much information a set of classical bits can store (calculating and reasoning about quantum bits is a separate learning goal). An *n*-bit number has two choices for each bit, giving it 2^n unique values, as addressed by HS.S.CP.8. It only stores one of those values at any point in time.

LEARNING OUTCOME

n classical bits can hold one of 2^n possible values.

(HS.Math.4b.1)

STANDARDS ALIGNMENT

Common Core State Standard:

- (HS.S.CP.8) Apply the general Multiplication Rule in a uniform probability model, P(A and B) = P(A)P(B|A) = P(B)P(A|B), and interpret the answer in terms of the model.

COURSE

- AP Statistics

HS Mathematics 4b.2

QUBITS

DESCRIPTION

A set of n bits has two possible choices for each bit, giving it 2^n unique values, as addressed by HS.S.CP.8. A set of n qubits can store a superposition of any or all of these 2^n possible values, with each value having a given probability of being measured. Bringing in the storage format (vectors) makes this learning goal dependent on the quantum state vector learning goals (LG.2.1 and LG.2.2).

LEARNING OUTCOME

n quantum bits can hold a superposition of 2^n possible states, with each state having a certain probability of being measured.

(HS.Math.4b.2)

STANDARDS ALIGNMENT

Common Core State Standard:

- (HS.S.CP.9) Use permutations and combinations to compute probabilities of compound events and solve problems.

COURSE

- AP Statistics

HS Mathematics 4c.1

QUBITS

DESCRIPTION

When qubits are entangled, the probability of their combined outcomes is no longer always the product of the probability of the independent outcomes. This matches with the definition of independent probabilities as well as determining whether or not probabilities are independent. Independence of probabilities determines whether or not multiple qubits are entangled.

LEARNING OUTCOME

Measurement outcomes of entangled quantum bits are not all independent.

(HS.Math.4c.1)

STANDARDS ALIGNMENT

Common Core State Standard:

- (HS.S.CP.2) Understand that two events A and B are independent if the probability of A and B occurring together is the product of their probabilities, and use this characterization to determine if they are independent.

COURSE

- Geometry
- Algebra

5. ENTANGLEMENT

KEY CONCEPT

Entanglement, an inseparable relationship between multiple qubits, is a key property of quantum systems necessary for obtaining a quantum advantage in most QIS applications.

a. When multiple quantum systems in superposition are entangled, their measurement outcomes are correlated. Entanglement can cause correlations that are different from what is possible in a classical system.

b. An entangled quantum system of multiple qubits cannot be described solely by specifying an individual quantum state for each qubit.

c. Quantum technologies rely on entanglement in different ways. When a fragile entangled state is maintained, a computational advantage can be realized. The extreme sensitivity of entangled states, however, can enhance sensing and communication.

6. COHERENCE/DECOHERENCE

KEY CONCEPT

For quantum information applications to be successfully completed, fragile quantum states must be preserved, or kept **coherent**.

a. Decoherence erodes superposition and entanglement through undesired interaction with the surrounding environment. Uncontrolled radiation, including light, vibration, heat, or magnetic fields, can all cause decoherence.

b. Some types of qubits are inherently isolated, whereas others require carefully engineered materials to maintain their coherence.

c. High decoherence rates limit the length and complexity of quantum computations; implementing methods that correct errors can mitigate this issue.

7. QUANTUM COMPUTING

KEY CONCEPT

Quantum computers, which use qubits and quantum operations, will solve certain complex computational problems more efficiently than classical computers.

a. Qubits can represent information compactly; more information can be stored and processed using 100 qubits than with the largest conceivable classical supercomputer.

b. Quantum data can be kept in a superposition of exponentially many classical states during processing, giving quantum computers a significant speed advantage for certain computations such as factoring large numbers (exponential speed-up) and performing searches (quadratic speed-up). However, there is no speed advantage for many other types of computations.

c. A fault-tolerant quantum computer corrects all errors that occur during quantum computation, including those arising from decoherence, but error correction requires significantly more resources than the original computation.

HS Mathematics 7a.1

QUANTUM COMPUTING

DESCRIPTION

The amount of information stored in classical bits and quantum bits is described LG.4.2 and LG.4.3. This learning goal does the final comparison between the amount of information that can be stored in a classical computer and on a quantum computer. Therefore, it engages the same mathematical concepts as the previous two learning goals.

LEARNING OUTCOME

More information can be stored in a quantum computer than a classical computer

(HS.Math.7a.1)

STANDARDS ALIGNMENT

Common Core State Standard:

- (HS.S.CP.9) Use permutations and combinations to compute probabilities of compound events and solve problems.

COURSE

- AP Statistics

HS Mathematics 7b.1

QUANTUM COMPUTING

DESCRIPTION

The amount of time it takes to run programs that solve computational problems on classical computers and quantum computers are compared by plotting linear, polynomial (of 2nd or higher degree), and exponential functions, where the graph displays the problem size vs. the running time of the program. Students can use this to understand the speed advantage of quantum computing. They can see that there are crossover points between classical computation and quantum computation - quantum computation starts at a higher point (slower), but as the problem size increases, its running time grows more slowly. Therefore, at some point, there is a crossover between the running time of quantum and classical computation.

LEARNING OUTCOME

Quantum algorithms can solve certain computational problems in polynomial time that require exponential time for classical computers

(HS.Math.7b.1)

STANDARDS ALIGNMENT

Common Core State Standard:

- (HS.F.LE.3) Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function.

COURSE

- Algebra 1
- Algebra 2

HS Mathematics 7b.2

QUANTUM COMPUTING

DESCRIPTION

One of the most prominent algorithms for quantum computers applies to the area of cryptography. Students can explore the basis of cryptography by exploring the difference in runtime between calculating a function and its inverse. When a function is easy to compute, but has an inverse that is hard to compute, it is called a one-way function. One-way functions play a fundamental role in many cryptographic applications. An example of a one-way function is the operation of multiplying prime numbers. On a classical computer, computing a product of two numbers is easy, but factoring a product of prime numbers is extremely difficult. Shor's algorithm, however, factors much more quickly than a classical computer, allowing fast decryption (once a quantum computer large enough to solve it is built) of commonly used types of public-key cryptography. Students can explore different one-way functions, their runtimes, and how quantum algorithms will even out the encryption vs decryption runtimes.

LEARNING OUTCOME

Quantum algorithms can break certain types of classical cryptography, which rely on the disparity between time to calculate a function's output vs its inverse

(HS.Math.7b.2)

STANDARDS ALIGNMENT

Common Core State Standard:

- (HS.F.LE.3) Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function.

COURSE

- Algebra 1
- Algebra 2

HS Mathematics 7c.1

QUANTUM COMPUTING

DESCRIPTION

Because quantum computers have errors so frequently, they require error mitigation in the form of fault tolerance. Without needing to understand exactly how they work, students can get a sense of one of the costs of fault tolerance: extra quantum bits. Depending on the error rate (per gate operation), the number of gates, and the number of logical qubits desired, the calculation for the number of physical qubits necessary to perform error-tolerant computation is the following:

$$O\left(q\left(log\frac{kq}{p_f}\right)^2\right)$$

where p_f is the overall probability of failure (error rate), k is the number of gates, and q is the number of logical qubits. Students can graph the number of physical qubits necessary either as the error rate changes, the number of gate operations changes, and/or the number of logical qubits changes.

LEARNING OUTCOME

Fault tolerance requires extra quantum bits to protect computation

(HS.Math.7c.1)

STANDARDS ALIGNMENT

Common Core State Standard:

- (HS.A-REI.11) Explain why the x-coordinates of the points where the graphs of the equations y = f(x) and y = g(x) intersect are the solutions of the equation f(x) = g(x); find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where f(x) and/or g(x) are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.

COURSE

- Algebra 1
- Algebra 2

8. QUANTUM COMMUNICATION

KEY CONCEPT

Quantum communication uses entanglement or a transmission channel, such as optical fiber, to transfer quantum information between different locations.

a. Quantum teleportation is a protocol that uses entanglement to destroy quantum information at one location and recreate it at a second site, without transferring physical qubits.

b. Quantum cryptography enhances privacy based on quantum physics principles and cannot be circumvented. Due to the fragility of quantum systems, an eavesdropper's interloping measurement will almost always be detected.

9. QUANTUM SENSORS

KEY CONCEPT

Quantum sensing uses quantum states to detect and measure physical properties with the highest precision allowed by quantum mechanics.

a. The Heisenberg uncertainty principle describes a fundamental limit in simultaneously measuring two specific, separate attributes. “Squeezing” deliberately sacrifices the certainty of measuring one attribute in order to achieve higher precision in measuring the other attribute; for example, squeezing is used in LIGO to improve sensitivity to gravitational waves.

b. Quantum sensors take advantage of the fact that physical qubits are extremely sensitive to their surroundings. The same fragility that leads to rapid decoherence enables precise sensors. Examples include magnetometers, single-photon detectors, and atomic clocks for improvements in medical imaging and navigation, position, and timing.

c. Quantum sensing has vastly improved the precision and accuracy of measurements of fundamental constants, freeing the International System of Units from its dependence on one-of-a-kind artifacts. Measurement units are now defined through these fundamental constants, like the speed of light and Planck’s constant.

HS Mathematics 9a.1

QUANTUM SENSORS

DESCRIPTION

There is an element of randomness in all measurement outcomes (LG.3.1 and LG.3.2). If we measure the position or momentum of a particle in motion, there will be a spread of possible outcomes that can be measured by the standard deviation (HS.SD.ID.2). The Heisenberg uncertainty principle describes mathematically how small we can make the uncertainties in the position and momentum. It says that multiplying the standard deviation for the position measurement and the standard deviation for the momentum measurement is always bigger than a certain fixed number. Students can explore the consequences of the uncertainty principle by interpreting the product of the two uncertainties as the area of a rectangle of fixed area. “Squeezing” can also be discussed by manipulating the length and width of this rectangle.

LEARNING OUTCOME

In quantum systems, there is a tradeoff in measurement precision between two attributes like position and momentum.

(HS.Math.9a.1)

STANDARDS ALIGNMENT

Common Core State Standard:

- (HS.SD.ID.2) Use statistics appropriate to the shape of the data distribution to compare center (median, mean) and spread (interquartile range, standard deviation) of two or more different data sets.

COURSE

- Algebra 1

CONNECTIONS BETWEEN QIS KEY CONCEPTS & CCSS

Below, you will find a summary of the topics in the QIS framework and potential links to the Common Core State Standards for Mathematics. Please note that the links to CCSS are forward-looking and meant to serve as conversation starters for the development of curricular activities. This framework is an invitation for collaboration between educators and researchers for developing these curricular activities and lesson plans.

QIS Topic / Concept	QIS Learning Goal	CCSS Connection	Course
1. Quantum Information Science	-	-	-
2. Quantum State	HS.Math.2a.1	HS.N.VM.1	Pre-calculus
	HS.Math.2a.2	HS.N.VM.5	Pre-calculus
	HS.Math.2b1.1	HS.S.MD.3	AP Statistics
	HS.Math.2b1.2	HS.S.CP.2	Algebra or Geometry
	HS.Math.2b2.1	HS.N.VM.11	Pre-calculus
	HS.Math.2b2.2	HS.N.CN.1	Algebra 2
	HS.Math.2b2.3	HS.N.CN.2	Pre-calculus
3. Measurement	HS.Math.3b.1	7.SP.6	Grade 7 Math
	HS.Math.3c.1	HS.S.IC	Algebra 2
4. Qubit	HS.Math.4a.1	7.SP.5	Grade 7 Math
	HS.Math.4b.1	HS.S.CP.8	AP Statistics
	HS.Math.4b.2	HS.S.CP.9	AP Statistics
	HS.Math.4c.1	HS.S.CP.2	Geometry or Algebra
5. Entanglement	-	-	-
6. Coherence / Decoherence	-	-	-
7. Quantum Computing	HS.Math.7a.1	HS.S.CP.9	AP Statistics
	HS.Math.7b.1	HS.F.LE.3	Algebra 1 or Algebra 2
	HS.Math.7b.2	HS.F.LE.3	Algebra 1 or Algebra 2
	HS.Math.7c.1	HS.A-REI.11	Algebra 1 or Algebra 2
8. Quantum Communication	-	-	-
9. Quantum Sensing	HS.Math.9a.1	HS.SD.ID.2	Algebra 1

GLOSSARY

QUANTUM STATE:
The mathematical representation of the properties of a physical system, such as an atom.

WAVEFUNCTION:
A mathematical description of the quantum state, from which measurement probabilities can be calculated.

SUPERPOSITION:
A linear combination of quantum states. A superposition of atomic orbitals is known as a hybridized orbital.

BIT:
A binary digit (bit) is the smallest unit of information, which can take the value “0” or “1”.

QUBIT:
A quantum bit (qubit) is the smallest unit of quantum information, which can be in the quantum state “0” or “1” or a linear combination of them.

ENTANGLEMENT:
An inseparable relationship that can exist between quantum objects or qubits.

DECOHERENCE:
The process by which superposition and entangled quantum states are corrupted by interactions with the surrounding environment.

CLASSICAL COMPUTER:
A device that processes information by performing operations on bits. This term generally refers to the traditional computers used today.

QUANTUM COMPUTER:
A device that processes quantum information by performing operations on quantum bits (qubits).

SENSOR:
A device or process that provides information about the state of an object or the environment (e.g., a magnetic field).